LAWS OF FORM

G. SPENCER-BROWN

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THE FORM

We take as given the idea of distinction and the idea of indication, and that we cannot make an indication without drawing a distinction. We take, therefore, the form of distinction for the form.

Definition

Distinction is perfect continence.

That is to say, a distinction is drawn by arranging a boundary with separate sides so that a point on one side cannot reach the other side without crossing the boundary. For example, in a plane space a circle draws a distinction.

Once a distinction is drawn, the spaces, states, or contents on each side of the boundary, being distinct, can be indicated.

There can be no distinction without motive, and there can be no motive unless contents are seen to differ in value.

If a content is of value, a name can be taken to indicate this value.

Thus the calling of the name can be identified with the value of the content.

Axiom 1. The law of calling

The value of a call made again is the value of the call.

That is to say, if a name is called and then is called again, the value indicated by the two calls taken together is the value indicated by one of them.

That is to say, for any name, to recall is to call.

Equally, if the content is of value, a motive or an intention or instruction to cross the boundary into the content can be taken to indicate this value.

Thus, also, the crossing of the boundary can be identified with the value of the content.

Axiom 2. The law of crossing

The value of a crossing made again is not the value of the crossing.

That is to say, if it is intended to cross a boundary and then it is intended to cross it again, the value indicated by the two intentions taken together is the value indicated by none of them.

That is to say, for any boundary, to recross is not to cross.

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FORMS TAKEN OUT OF THE FORM

Construction

Draw a distinction.

Content

Call it the first distinction.

Call the space in which it is drawn the space severed or cloven by the distinction.

Call the parts of the space shaped by the severance or cleft the sides of the distinction or, alternatively, the spaces, states, or contents distinguished by the distinction.

Intent

Let any mark, token, or sign be taken **in any way with or** with regard to the distinction as a signal.

Call the use of any signal its intent.

First canon. Convention of intention

Let the intent of a signal be limited to the use allowed to it.

Call this the convention of intention. In general, *what is not allowed is forbidden.*

Knowledge

Let a state distinguished by the distinction be marked with a mark

⌉

of distinction.

Let the state be known by the mark.

Call the state the marked state.

Form

Call the space cloven by any distinction, together with the entire content of the space, the form of the distinction.

Call the form of the first distinction the form.

Name

Let there be a form distinct from the form.

Let the mark of distinction be copied out of the form into such another form.

Call any such copy of the mark a token of the mark.

Let any token of the mark be called as a name of the marked state.

Let the name indicate the state.

Arrangement

Call the form of a number of tokens considered with regard to one another (that is to say, considered in the same form) an arrangement.

Expression

Call any arrangement intended as an indicator an expression.

Value

Call a state indicated by an expression the value of the expression.

Equivalence

Call expressions of the same value equivalent.

Let a sign

$$=$$

of equivalence be written between equivalent expressions.

Now, by axiom 1,

$$\overline{}| \; \overline{}| = \overline{}| \, .$$

Call this the form of condensation.

Instruction

Call the state not marked with the mark the unmarked state.

Let each token of the mark be seen to cleave the space into which it is copied. That is to say, let each token be a distinction in its own form.

Call the concave side of a token its inside.

Let any token be intended as an instruction to cross the boundary of the first distinction.

Let the crossing be from the state indicated on the inside of the token.

Let the crossing be to the state indicated by the token.

Let a space with no token indicate the unmarked state.

Now, by axiom 2,

$$\overline{\overline{}|}| = \quad .$$

Call this the form of cancellation.

Equation

Call an **indication of equivalent expressions an equation.**

Primitive equation

Call the form of condensation a primitive equation.

Call the form of cancellation a primitive equation.

Let there be no other primitive equation.

Simple expression

Note that the three forms of arrangement, ⌉ ⌉ , ⌉⌉ , ⌉ , and the one absence of form, , taken from the primitive equations are all, by convention, expressions.

Call any expression consisting of an empty token simple.

Call any expression consisting of an empty space simple.

Let there be no other simple expression.

Operation

We now see that if a state can be indicated by using a token as a name it can be indicated by using the token as an instruction subject to convention. Any token may be taken, therefore, to be an instruction for the operation of an intention, and may itself be given a name

cross

to indicate what the intention is.

Relation

Having decided that the form of every token called cross is to be perfectly continent, we have allowed only one kind of relation between crosses: continence.

Let the intent of this relation be restricted so that a cross is said to contain what is on its inside and not to contain what is not on its inside.

Depth

In an arrangement *a* standing in a space *s*, call the number *n* of crosses that must be crossed to reach a space s_n from *s* the depth of s_n with regard to *s*.

Call a space reached by the greatest number of inwards crossings from *s* a deepest space in *a*.

Call the space reached by no crossing from *s* the shallowest space in *a*.

Thus

$$s_0 = s.$$

Let any cross standing in any space in a cross *c* be said to be contained in *c*.

Let any cross standing in the shallowest space in *c* be said to stand under, or to be covered by, *c*.

Unwritten cross

Suppose any s_0 to be surrounded by an unwritten cross.

Call the crosses standing under any cross *c*, written or unwritten, the crosses pervaded by the shallowest space in *c*.

Pervasive space

Let any given space s_n be said to pervade any arrangement in which s_n is the shallowest space.

Call the space *s* pervading an arrangement *a*, whether or not *a* is the only arrangement pervaded by *s*, the pervasive space of *a*.

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RE-ENTRY INTO THE FORM

The conception of the form lies in the desire to distinguish.

Granted this desire, we cannot escape the form, although we can see it any way we please.

The calculus of indications is a way of regarding the form.

We can see the calculus by the form and the form in the calculus unaided and unhindered by the intervention of laws, initials, theorems, or consequences.

The experiments below illustrate one of the indefinite number of possible ways of doing this.

We may note that in these experiments the sign

$$=$$

may stand for the words

is confused with.

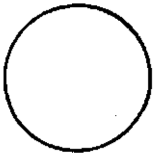
We may also note that the sides of each distinction experimentally drawn have two kinds of reference.

The first, or explicit, reference is to the value of a side, according to how it is marked.

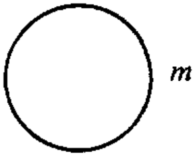
The second, or implicit, reference is to an outside observer. That is to say, the outside is the side from which a distinction is supposed to be seen.

First experiment

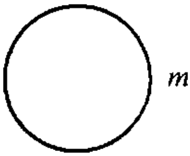
In a plane space, **draw a** circle.



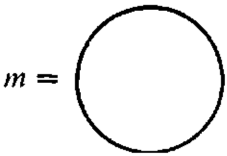
Let a mark *m* indicate the outside of the **circumference.**



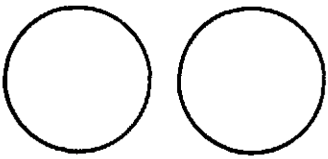
Let **no mark** indicate the inside of the **circumference.**



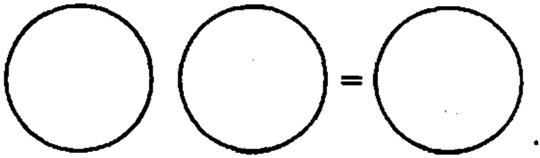
Let the mark *m* be a circle.



Re-enter the mark into the form of the circle.

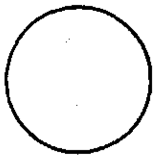


Now the circle and the mark cannot (in respect of their relevant properties) be distinguished, and so

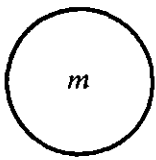


Second experiment

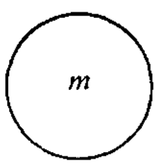
In a plane space, draw a circle.



Let a mark *m* indicate the inside of the circumference.



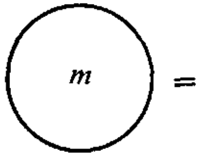
Let no mark indicate the outside of the circumference.



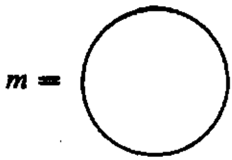
Let the value of a mark be its value to the space in which it stands. That is to say, let the value of a mark be to the space outside the mark.

Now the space outside the circumference is unmarked.

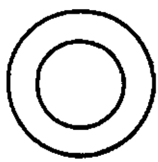
Therefore, by valuation,



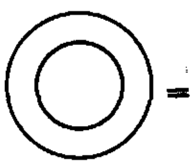
Let the mark m be a circle.



Re-enter the mark into the form of the circle

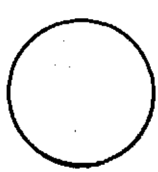


Now, by valuation,

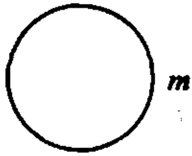


Third experiment

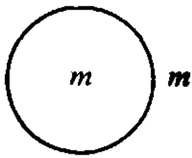
In a plane space, draw a circle.



Let a mark *m* indicate the outside of the circumference.

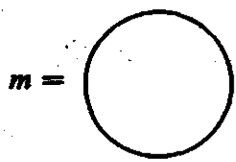


Let a similar mark *m* indicate the inside of the circumference.

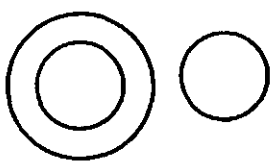


Now, since a mark *m* indicates both sides of the circumference, they cannot, in respect of value, be distinguished.

Again let the mark *m* be a circle.



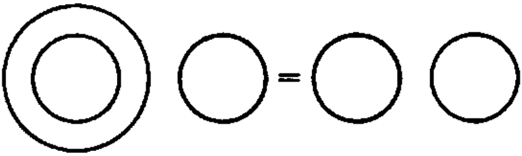
Re-enter the mark into the form of the circle.



Now, because of identical markings, the original circle cannot distinguish different values.

Therefore, it is not, in this respect, a distinction.

Therefore it may be deleted without loss or gain to the space in which it stands.



But we found in the first experiment that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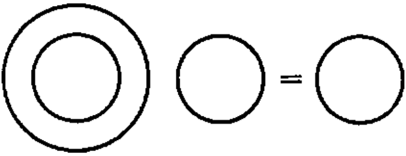
Therefore,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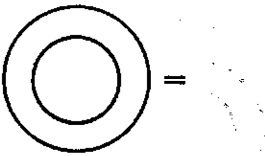
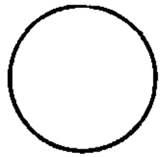
and this is not inconsistent with the finding of the second experiment that


,

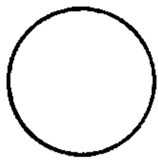
since we have done here in two steps which was done there in one.

Fourth experiment

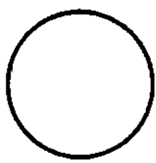
In a plane space, **draw** a circle.



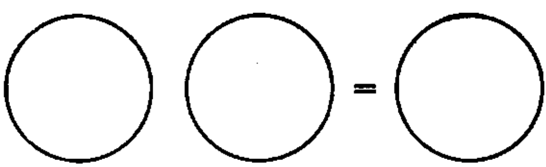
Let the outside of the circumference be **unmarked.**



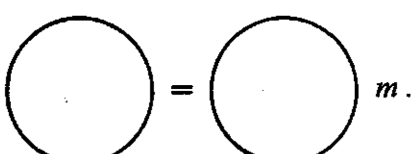
Let the inside of the circumference be **unmarked.**



But we saw in the **first** experiment that



and that therefore, by reversing the purifying procedure **there,**



The value of a circumference to the space outside must be, therefore, the value of the mark, since the mark now distinguishes this space.

An observer, since he distinguishes the space he occupies, is also a mark.

In the experiments above, imagine the circles to be forms and their circumferences to be the distinctions shaping the spaces of these forms.

In this conception a distinction drawn in any space is a mark distinguishing the space. Equally and conversely, any mark in a space draws a distinction.

We see now that the first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical.

NOTES

Chapter 1

Although it says somewhat more, all that the reader needs to take with him from Chapter 1 are the definition of distinction as a form of closure, and the two axioms which rest with this definition.

Chapter 2

It may be helpful at this stage to realize that the primary form of mathematical communication is not description, but injunction. In this respect it is comparable with practical art forms like cookery, in which the taste of a cake, although literally indescribable, can be conveyed to a reader in the form of a set of injunctions called a recipe. Music is a similar art form, the composer does not even attempt to describe the set of sounds he has in mind, much less the set of feelings occasioned through them, but writes down a set of commands which, if they are obeyed by the reader, can result in a reproduction, to the reader, of the composer's original experience.

Where Wittgenstein says [4, proposition 7]

> whereof one cannot speak,
> thereof one must be silent

he seems to be considering descriptive speech only. He notes elsewhere that the mathematician, descriptively speaking, says nothing. The same may be said of the composer, who, if he were to attempt a *description* (i.e. a limitation) of the set of ecstasies apparent *through* (i.e. unlimited by) his *composition*, would fail miserably and necessarily. But neither the composer nor the mathematician must, for this reason, be silent.

In his introduction to the *Tractatus*, Russell expresses what thus seems to be a justifiable doubt in respect of the rightness of Wittgenstein's last proposition when he says [p 22]

> what causes hesitation is the fact that, after all, Mr Wittgenstein manages to say a good deal about what cannot be said, thus suggesting to the sceptical reader that possibly there may be some loophole through a hierarchy of languages, or by some other exit.

The exit, as we have seen it here, is evident in the injunctive faculty of language.

Even natural science appears to be more dependent upon injunction than we are usually prepared to admit. The professional initiation of the man of science consists not so much in reading the proper textbooks, as in obeying injunctions such as 'look down that microscope'. But it is not out of order for men of science, having looked down the microscope, now to describe to each other, and to discuss amongst themselves, what they have seen, and to write papers and textbooks describing it. Similarly, it is not out of order for mathematicians, each having obeyed a given set of injunctions, to describe to each other, and to discuss amongst themselves, what they have seen, and to write papers and textbooks describing it. But in each case, the description is dependent upon, and secondary to, the set of injunctions having been obeyed first.

When we attempt to realize a piece of music composed by another person, we do so by *illustrating*, to ourselves, with a musical instrument of some kind, the composer's commands. Similarly, if we are to realize a piece of mathematics, we must find a way of illustrating, to ourselves, the commands of the mathematician. The normal way to do this is with some kind of scorer and a flat scorable surface, for example a finger and a tide-flattened stretch of sand, or a pencil and a piece of paper. Taking such an aid to illustration, we may now begin to carry out the commands in Chapter 2.

First we may illustrate a form, such as a circle or near-circle. A flat piece of paper, being itself illustrative of a plane surface, is a useful mathematical instrument for this purpose, since we happen to know that a circle in such a space does in fact draw a distinction. (If, for example, we had chosen to write upon the surface of a torus, the circle might not have drawn a distinction.)

When we come to the injunction

let there be a form distinct from the form

we can illustrate it by taking a fresh piece of paper (or another stretch of sand). Now, in this separate form, we may illustrate the command

let the mark of distinction be copied
out of the form into such another form.

It is not necessary for the reader to confine his illustrations to the commands in the text. He may wander at will, inventing his own illustrations, either consistent or inconsistent with the textual commands. Only thus, by his own explorations, will he come to see distinctly the bounds or laws of the world from which the mathematician is speaking. Similarly, if the reader does not follow the argument at any point, it is never necessary for him to remain stuck at that point until he sees how to proceed. We cannot fully understand the beginning of anything until we see the end. What the mathematician aims to do is to give a complete picture, the order *of what* he presents being essential, the order *in which* he presents it being to some degree arbitrary. The reader may quite legitimately change the arbitrary order as he pleases.

We may distinguish, in the essential order, *commands*, which call something into being, conjure up some order of being, call to order, and which are usually carried in permissive forms such as

let there be so-and-so,

or occasionally in more specifically active forms like

drop a perpendicular;

names, given to be used as reference points or tokens; in relation with the operation of *instructions*, which are designed to take effect within whatever universe has already been commanded or called to order. The institution or ceremony of naming is usually carried in the form

call so-and-so such-and-such,

and the call may be transmitted in both directions, as with the sign =, so that by calling so-and-so such-and-such we may also call such-and-such so-and-so. Naming may thus be considered to be without direction, or, alternatively, pan-directional. By contrast, instruction is directional, in that it demands a crossing from a state or condition, with its own name, to a different state or condition, with another name, such that the name of the former may not be called as a name of the latter.

The more important structures of command are sometimes called canons. They are the ways in which the guiding injunctions appear to group themselves in constellations, and are thus by no means independent of each other. A canon bears the distinction of being outside (i.e. describing) the system under construction, but a command to construct (e.g. 'draw a distinction'), even though it may be of central importance, is not a canon. A canon is an order, or set of orders, to permit or allow, but not to construct or create.

The instructions which are to take effect, within the creation and its permission, must be distinguished as those in the actual text of calculation, designated by the constants or *operators* of the calculus, and those in the context, which may themselves be instructions to name something with a particular name so that it can be referred to again without redescription.

Later on (Chapter 4) we shall come to consider what we call the proofs or justifications of certain statements. What we shall be showing, here, is that such statements are implicit in, or follow from, or are permitted by, the canons or standing orders hitherto convened or called to presence. Thus, in the structure of a proof, we shall find injunctions of the form consider such-and-such,

suppose so-and-so,

which are not commands, but *invitations* or *directions* to a *way* in which the implication can be clearly and wholly followed.

In conceiving the calculus of indications, we begin at a point of such *degeneracy* as to find that the ideas of description, indication, name, and instruction can amount to the same thing. It is of some importance for the reader to realize this for himself, or he will find it difficult to understand (although he may follow) the argument (p 5) leading to the second primitive equation.

In the command

let the crossing be to the
state indicated by the token

we at once make the token doubly meaningful, first as an instruction to cross, secondly as an indicator (and thus a name) of where the crossing has taken us. It was an open question, before obeying this command, whether the token would carry an indication at all. But the command determines without ambiguity the state to which the crossing is made and thus, without ambiguity, the indication which the token will henceforth carry.

This double carry of name-with-instruction and instruction-with-name is usually referred to (in the language of mathematics) as a structure in which ideas or meanings *degenerate*. We may also refer to it (in the language of psychology) as a place where the ideas *condense* in one symbol. It is this condensation which gives the symbol its *power*. For in mathematics, as in other disciplines, the power of a system resides in its elegance (literally, its capacity to pick out or elect), which is achieved by condensing as much as is needed into as little as is needed, and so making that little as free from irrelevance (or from elaboration) as is allowed by the necessity of writing it out and reading it in with ease and without error.

We may now helpfully distinguish between an elegance in the calculus, which can make it easy to use, and an elegance in the descriptive context, which can make it hard to follow. We are accustomed, in ordinary life, to having indications of what to do confirmed in several different ways, and when presented with an injunction, however clear and unambiguous, which, stripped to its bare minimum, indicates what to do once and in one way only, we might refuse it. (We may consider how far, in ordinary life, we must observe the spirit rather than the letter of an injunction, and must develop the habitual capacity to interpret any injunction we receive by screening it against other indications of what we ought to do. In mathematics we have to unlearn this habit in favour of accepting an injunction literally and at once. This is why an author of mathematics must take such great pains to make his injunctions mutually permissive. Otherwise these pains, which rightly rest with the author, will fall with sickening import upon the reader, who, by virtue of his relationship with respect to the author, may be in no position to accept them.)

The second of the two primitive equations of the primary arithmetic can be derived less elegantly, but in a way that is possibly easier to follow, by allowing substitution prematurely.

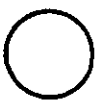
Suppose we indicate the marked state by a token *m*, and, as before, let the absence of a token indicate the unmarked state.

Let a bracket round any indicator indicate, in the space outside the bracket, the state other than that indicated inside the bracket.

Thus

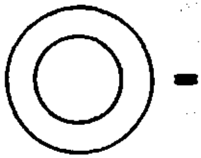
=

and



= *m*.

Substituting, we find



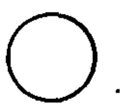
which is the second primitive equation.

The condition that one of the primary states shall be nameless is mandatory for this elimination.

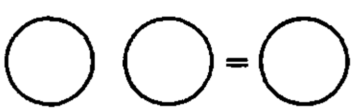
The first primitive equation can also be derived a different way.

Imagine a blind animal able only to distinguish inside from outside. A space with what appears to us as a number of distinct insides and one outside, such as

will appear to it, upon exploration, to be indistinguishable from



The ideas described in the text at this point do not go beyond what this animal can find out for itself, and so in its world, such as it is,



We may note that even if this animal can count its crossings, it still will not be able to distinguish two divisions from one, although it will now have an alternative way of distinguishing inside from outside which no longer depends on knowing which is which.

Reconsidering the first command,

draw a distinction,

we note that it may equally well be expressed in such ways as

let there be a distinction,

find a distinction,

see a distinction,

describe a distinction,

define a distinction,

or

let a distinction be drawn,

for we have here reached a place so primitive that active and passive, as well as a number of other more peripheral opposites, have long since condensed together, and almost any form of words will suggest more categories than there really are.

Chapter 3

The hypothesis of simplification is the first *overt* convention that is put to use before it has been justified. But it has a precursor in the injunction 'let a state indicated by an expression be the value of the expression' in the last chapter, which allows value to an expression only in case not less and not more than one state is indicated by the expression. The use of both the injunction and the convention are eventually justified in the theorems of representation. Other cases of delayed justification will be found later, a notable example being theorem 16.

We may ask why we do not justify such a convention at once when it is given. The answer, in most cases, is that the justification (although valid) would be meaningless until we had first become acquainted with the *use* of the principle which requires justifying. In other words, before we can reasonably justify a deep lying principle, we first need to be familiar with how it works.

We might suppose this practice of deferred justification to be operative elsewhere. It is a notable fact that in mathematics very few *useful* theorems remain unproved. By 'useful' I do not necessarily mean with practical application outside mathematics. A theorem can be useful mathematically, for example to justify another theorem.

One of the most 'useless' theorems in mathematics is Goldbach's conjecture. We do not frequently find ourselves saying 'if only we knew that every even number greater than 2 could be represented as a sum of two prime numbers, we should be able to show that . . .' D J Spencer Brown, in a private communication, suggested that their apparent uselessness is not exactly a reason why such theorems cannot be proved, but is a reason for supposing that if a valid proof were given today, nobody would recognize it as such, since nobody is yet *familiar* with the *ground* on which such a proof would rest. I shall have more to say about this in the notes to Chapters 8 and 11.

Chapter 4

In all mathematics it becomes apparent, at some stage, that we have for some time been following a rule without being consciously aware of the fact. This might be described as the use of a *covert* convention. A recognizable aspect of the advancement of mathematics consists in the advancement of the consciousness of what we are doing, whereby the covert becomes overt. Mathematics is in this respect psychedelic.

The nearer we are to the beginning of what we set out to achieve, the more likely we are to find, there, procedures which have been adopted without comment. Their use can be considered as the presence of an arrangement in the absence of an agreement. For example, in the statement and proof of theorem 1 it is arranged (although not agreed) that we shall write on a plane surface. If we write on the surface of a torus the theorem is not true. (Or to make it true, we must be more explicit.)

The fact that men have for centuries used a plane surface for writing means that, at this point in the text, both author and reader are ready to be conned into the assumption of a plane writing surface without question. But, like any other assumption, it is not unquestionable, and the fact that we can question it here means that we can question it elsewhere. In fact we have found a common but hitherto unspoken assumption underlying what is written in mathematics, notably a plane surface (more generally, a surface of genus 0, although we shall see later (pp 102 sq) that this further generalization forces us to recognize another hitherto silent assumption). Moreover, it is now evident that if a different surface is used, what is written on it, although identical in marking, may be not identical in meaning.

In general there is an order of precedence amongst theorems, so that theorems which can be proved more easily with the help of other theorems are placed so as to be proved after such other theorems. This order is not rigid. For example, having proved theorem 3, we use what we found in the proof to prove theorem 4. But theorems 3 and 4 are symmetrical, their order depending only on whether we wish to proceed from simplicity to complexity or from complexity to simplicity. The reader might try, if he wishes, to prove theorem 4 first without the aid of theorem 3, after which he will be able to prove theorem 3 analogously to the way theorem 4 is proved in the text.

It will be observed that the symbolic representation of theorem 8 is less strong than the theorem itself. The theorem is consistent with

$$\overline{\overline{p}|\; pq}| = \qquad ,$$

whereas we prove the weaker version

$$\overline{\overline{p}|\; p}| =$$

The stronger version is plainly true, but we shall find that we are able to demonstrate it as a consequence in the algebra. We therefore prove, and use as the first algebraic initial, the weaker version.

In theorem 9 we see the difference between our use of the verb *divide* and our use of the verb *cleave*. Any division of a space results in *otherwise indistinguishable divisions of a state*, which are all at the same level, whereas a severance or cleavage shapes *distinguishable states*, which are at different levels.

An idea of the relative strengths of severance and division may be gathered from the fact that the rule of number is sufficient to unify a divided space, but not to void a cloven space.

Chapter 5

In eliciting rules for algebraic manipulation the text explicitly refers to the existence of systems of calculation other than the system described. This reference is both deliberate and inessential. It marks the level at which these systems are usually fitted out with their false, or truncated, or postulated, origins.

It is deliberate to inform the reader that, in the system of calculation we are building, we are not departing from the basic methods of other systems. Thus what we arrive at, in the end, will serve to elucidate them, as well as to fit them with their true origin. But, at the same time, it is important for the reader to see that the reference to other systems is inessential to the development of the argument in the text. For here it stands or falls on its own merit, dependent in no way for its validity upon agreement or disagreement with other systems. Thus rules 1 and 2, as can be seen from their justifications, say nothing that has not, in the text, already been said. They merely summarize the commands and instructions that will be relevant to the new kind of calculation we are about to undertake.

The replacement referred to in rule 2 is usually confined to independent variable expressions of simple (i.e. literal) form, and is in fact so confined in the text. But the greater licence granted by the rule is not devoid of significant application, if required.

Chapter 6

By the revelation and incorporation of its own origin, the primary algebra provides immediate access to the nature of the relationship between operators and operands. An operand in the algebra is merely a conjectured presence or absence of an operator.

This partial identity of operand and operator, which is not confined to Boolean algebras, can in fact be seen if we extend more familiar descriptions, although in these descriptions it is not so obvious. For example, we can find it by taking the Boolean operators ∨ (usually interpreted as the logical 'or', but here used purely mathematically) and . (usually interpreted as the logical 'and', but here again used purely mathematically), freeing their scope (as, by the principle of relevance, we may), freeing the order of the variables within their scope (as, by the same principle, we also may), and extrapolating mathematically to the case of no variable,

	...(*a b c*)	∨	.	(*a b*)	∨	.	(*a*)	∨	.	()	∨	.
permute	1 1 1	1	1	1 1	1	1	1	1	1		0	1
permute	1 1 0	1	0	1 0	1	0	0	0	0			
permute	1 0 0	1	0	0 0	0	0						
permute	0 0 0	0	0									

,

which shows quite plainly that we have no need of the arithmetical forms 0, 1 (or *z*, *u*, or *F*, *T*, etc), since we can equate them with ()∨ and (). respectively. We can now write a Boolean variable of the form *a*, *b*, etc wherever we conjecture the presence of one of these two fundamental particles, but are not sure (or don't care) which. The functional tables for ∨ and . of two variables thus become

(*a*	*b*)	∨	.
(()∨	()∨)	()∨	()∨
(()∨	().)	().	()∨
(().	().)	().	().

,

the permutation being assumed.

J1, J2 are not the only two initials which may be taken to determine the primary algebra. We see[11] from Huntington's fourth postulate-set that we could have used C5, C6.

[11] Edward V Huntington, *Trans. Amer. Math. Soc.*, 35 (1933) 280–5.

The demonstration of J1, J2 from C5, C6 is both difficult and tedious. This is evidently because we find two basic algebraic principles, in one of which a variable is transplanted in the expression, and in the other of which it is eliminated from it. Provided we keep these two principles apart, subsequent demonstrations are not difficult. If, as in Huntington's two equations, they are inter-mingled, then their subsequent unravelling can be difficult.

Our expression here of Huntington's equations in the form of C5, C6 is not in the form in which he originally expressed them. He was hampered by the crippling assumptions of order relevance and binary scope, with which we have not at any stage weakened the primary algebra. For this reason he found it necessary to give two more equations to complete the set. C5 and C6, considered as initials, are of interest chiefly because they employ only two distinct variables, whereas J1 and J2 employ three.

I had at first supposed the demonstration of C1 to be impossible from J1 and J2 as they stand. In 1965 a pupil, Mr John Dawes, produced a rather long proof to the contrary, so the following year I set the problem to my class as an exercise, and was rewarded with a most elegant demonstration by Mr D A Utting. I use Mr Utting's demonstration, slightly modified, in the text.

Although, superficially, it may look less efficient, it is, eventually, more natural and convenient to use names rather than numbers to identify the more important consequences, as indeed it is with theorems, since they do not in general form an ordered set.

In naming such consequences I have aimed to find what seems appropriate as a description of the named process, as it appears in the algebra, without doing violence to its arithmetical origin. In some places both the forms and the names are recognizably similar to those of other authors who have determined Boolean algebras. In most such cases hitherto, the commonly used name describes only one of the directions in which the step can be taken. What is called Boolean expansion is an example. In such a case, where the name is appropriate to the step as taken in one direction only, I have introduced an antonym for the other direction, and given a generic name to cover both. In other recognizable cases I have found what seems to me to be a more appropriate name, such as occultation for what Whitehead called[12] absorption. The occulting part of the expression is not so much absorbed in the remainder as eclipsed by it. This can be seen quite plainly in the arithmetic, or alternatively if the expression is illustrated with a Venn diagram. To the best of my knowledge, Peirce was the only previous author to recognize, as such, what I call position. He called[13] it erasure, thus again drawing attention to only one direction of application.

I do not suppose all the names will always stick. Familiarity tends to produce a kind of in-slang, often more appropriate, in its place, than what is deemed to be academically proper or seemly. For example, the engineering application of consequence 2 has produced the more homely 'breed' for 'regenerate', and 'revert' for 'degenerate', and it is of interest to note that the transformations of this consequence are immediate images of what Proclus called[14] πρόοδος and ἐπιστροφή, translated by Dodds into *procession* and *reversion.*

The fact that descriptive names such as 'transposition' and 'integration' are differently applied elsewhere in mathematics (and, indeed, elsewhere in this book) does not appear to be a reason for avoiding their use in the senses defined in this chapter. The deeper the level of investigation, the harder it becomes to find words strong enough to cover what is found there, and in all cases my use of language to describe primitive processes draws on a greater power of signification than is needed for its more superficial and specialized uses.

One of the most beautiful facts emerging from mathematical

[12] Alfred North Whitehead, *A treatise on universal algebra*, Vol. I, Cambridge, 1898, p 36.

[13] Charles Sanders Peirce, *Collected papers*, Vol. IV, Cambridge, Massachusetts, 1933, pp 13–18.

[14] *ΠΡΟΚΛΟΥ ΔΙΑΔΟΧΟΥ ΣΤΟΙΧΕΙΩΣΙΣ ΘΕΟΛΟΓΙΚΗ* with a translation by E R Dodds, 2nd edition, Oxford, 1963.

studies is this very potent relationship between the mathematical process and ordinary language. There seems to be no mathematical idea of any importance or profundity that is not mirrored, with an almost uncanny accuracy, in the common use of words, and this appears especially true when we consider words in their original, and sometimes long forgotten, senses.

The fact that a word may have different, but related, meanings at different, but related, levels of consideration does not normally render communication impossible. On the contrary, it is evident that communication of any but the most trivial ideas would be impossible without it.

Since at this point in the text the fundamental forms of mathematical communication are now practically complete, it may be a revealing exercise to retranslate into longhand some of the shorthand forms developed by application of the canon of contracting reference. For this purpose we take the statement and demonstration of consequence 9 (p 35). In words and figures it could run thus.

The ninth consequence, called crosstransposition, or C9 for short, may be stated as follows.

b cross *r* cross cross all *a*
cross *r* cross cross 2 *x* cross
r cross 2 *y* cross *r* cross 2
cross all

expresses the same value as

r cross *ab* cross all *rxy* cross 3.

When the step allowed by this equation is taken from the former to the latter expression, it is called to crosstranspose or collect, and when taken in reverse it is called to crosstranspose or distribute.

The equation can be demonstrated thus.

b cross *r* cross cross all *a*
cross *r* cross cross 2 *x* cross
r cross 2 *y* cross *r* cross 2
cross all

may be changed to

> *b* **cross** *r* **cross cross all** *a*
> cross *r* cross cross 2 *xy*
> **cross 2** *r* **cross 2 cross all**

by using C1, J2, and then C1 again. This in turn may be changed to

> *baxy* cross 2 *r* cross 2 cross
> all *rxy* cross 2 *r* cross 2
> cross 2

by C8 and then by applying C1 three times, etc.

We may observe that, in expressions, the mathematical language has become entirely visual, there is no proper spoken form, so that in reverbalizing it we must *encode* it in a form suitable for ordinary speech. Thus, although the mathematical form of an expression is clear, the reverbalized form is obscure.

The main difficulty in translating from the written to the verbal form comes from the fact that in mathematical writing we are free to mark the two dimensions of the plane, whereas in speech we can mark only the one dimension of time.

Much that is unnecessary and obstructive in mathematics today appears to be vestigial of this limitation of the spoken word. For example, in ordinary speech, to avoid direct reference to a plurality of dimensions, we have to fix the scope of constants such as 'and' and 'or', and this we can most conveniently do at the level of the first plural number. But to carry the fixation over into the written form is to fail to realize the freedom offered by an added dimension. This in turn can lead us to suppose that the binary scope of operators assumed for the convenience of representing them in one dimension is something of relevance to the actual form of their operation, which, in the case of simple operators even at the verbal level, it is not.

Chapter 7

In the description of theorem 14 'the constant' refers to the operative constant. There are two constants in the calculus, a mark or operator, and a blank or void. Reference to 'the constant' without qualification will usually be taken to denote the operator rather than the void.

Chapter 8

We have already distinguished, in the text, between demonstration and proof. In making this distinction, which appears quite natural, we see at once that a proof can never be justified in the same way as a demonstration. Whereas in a demonstration we can see that *the instructions already recorded are properly* obeyed, we cannot avail ourselves of this procedure in the case of a proof.

In a proof we are dealing in terms which are outside of the calculus, and thus not amenable to its instructions. In any attempt to render such proofs themselves subject to instruction, we succeed only at the cost of making another calculus, inside of which the original calculus is cradled, and outside of which we shall again see forms which are amenable to proof but not demonstration.

The validity of a proof thus rests not in our common motivation by a set of instructions, but in our common experience of a state of affairs. This experience usually includes the ability to reason which has been formalized in logic, but is not confined to it. Nearly all proofs, whether about a system containing numbers or not, use the common ability to compute, i.e. to count* in either direction, and ideas stemming from our experience of this ability.

It seems open to question why we regard the proof of a theorem as amounting to the same degree of certainty as the demonstration of a consequence. It is not a question which, at first sight, admits of an easy answer. If an answer is possible, it would seem to lie in the concept of experience. We gain experience of living representative processes, in particular of

* Although *count* rests on *putare* = prune, correct, (and hence) reckon, the word *reason* comes from *reri* = count, calculate, reckon. Thus the reasoning and computing activities of proof were originally considered as one. We may note further that *argue* is based on *arguere* = clarify (literally 'make silver'). We thus find a whole constellation of words to do with the process of *getting it right*.

argument and of counting forwards and backwards in units, and through this experience become quite certain, in our own minds, of the validity of using it to substantiate a proof. But since the procedures of the proof are not, themselves, yet codified in a calculus (although they may eventually become so), our certainty at this stage must be deemed to be intuitive. We can achieve a demonstration simply by following instructions, although we may be unfamiliar with the system in which the instructions are obeyed. But in proving a theorem, if we have not already codified the structure of the proof *in* the form of a calculus, we must at least be familiar with, or experienced in, whatever it is we take to be the *ground* of the proof, otherwise we shall not *see* it *as* a proof.

Another way of regarding the relationship between demonstration and proof, which adds support to the proposition that the degree of certainty of a proof is equal to that of a demonstration, is to consider it as the boundary dividing the state of proof from the state of demonstration. A demonstration, we remember, occurs inside the calculus, a proof outside. The boundary between them is thus a shared boundary, and is what is approached, in one or the other direction, according to whether we are demonstrating a consequence or proving a theorem. Thus consequences and theorems can be seen to bear to each other a fitting relationship.

But the boundary marking their relationship, although shared, is (like the existential boundary (see pp 124 sq)) seen from one side only, since if we know the ground on which a demonstration rests (i.e. provided we understand the formal, as distinct from the pragmatic, reasons for the initial equations we employ, and so do not have to postulate them), the demonstration can be seen as a proof by implication, although a proof is never seen as a demonstration. We observe, in fact, that demonstration bears the same relationship to proof as initial equation bears to axiom, but we should also note that the relationship is evident for arithmetic only, and is lost when we make the departure into algebra. This appears to be why algebras are commonly presented without axioms, in any proper sense of the word.

The fact that a proof is a way of making apparently obvious what was already latently so is of some mathematical interest. Although there are any number of distinct proofs of a given theorem, they can all, even so, be hard to find. In other words, we can set about trying to prove a theorem in a large number of wrong ways before coming across a right way.

Even the analogy of seeking something cannot, in this context, be quite right. For what we find, eventually, is something we have known, and may well have been consciously aware of, all along. Thus we are not, in this sense, seeking something that has ever been hidden. The idea of performing a search can be unhelpful, or even positively obstructive, since searches are in general organized to find something which has been previously hidden, and is thus not open to view.

In discovering a proof, we must do something more subtle than search. We must come to see the *relevance*, in respect of whatever statement it is we wish to justify, of some fact in full view, and of which, therefore, we are already constantly aware. Whereas we may know how to undertake a search for something we can *not* see, the subtlety of the technique of trying to 'find' something which we already *can* see may more easily escape our efforts.

This might be a helpful moment to introduce a distinction between following a course of argument and understanding it. I take understanding to be the experience of what is understood in a wider context. In this sense, we do not fully understand a theorem until we are able to contain it in a more general theorem. We can nevertheless follow its proof, in the sense of coming to see its evidence, without understanding it in the wider sense in which it may rest.

Following and understanding, like demonstrating and proving, are sometimes wrongly taken as synonymous. Very often a person is regarded as not understanding an argument, a process, a doctrine, when all that is certain is that he has not followed it. But his failure to follow may be quite deliberate, and may arise from the fact that he *has* understood what was presented to him, and does not follow it because he sees a shorter, or otherwise more acceptable, path, although he might not, yet, know how to communicate it.

Following may thus be associated particularly with doctrine, and doctrine demands an adherence to a particular way of saying or doing something. Understanding has to do with the fact that what ever is said or done can always be said or done a different way, and yet all ways remain the same.

Chapter 9

We observe that the idea of completeness cannot apply to a calculus as a whole, but only to a representation of one determination of it by another. What is questioned, in fact, is the completeness of an alternative form of expression.

The paragon of such an alternative is the algebraic representation of an arithmetic, although we do in fact find a more central case of it in the arithmetical representation of a form. In the latter case, as we see from the theorems of representation, the idea of completeness condenses with that of consistency. In the less central case, the two ideas come apart. Thus the most primitive example of completeness, in its pure form, is to be found in algebraic representation.

A fact to which Gödel drew attention [5] is that an algebra which includes representations of addition and multiplication *cannot* fully account for an arithmetic of the natural numbers in which these operations are taken as elementary. Thus, in number theory, although certain relationships can be proved, no algebra can be constructed in which all such relationships are demonstrable.

The advent of Gödel's theorem has never seemed to me to be a reason for despair, as some investigators have taken it to be, but rather an occasion for celebration, since it confirms what men of mathematics have found from experience, notably that ordinary arithmetic is a richer ground for investigation than ordinary algebra.

Chapter 10

It is usual to prove the independence of initial equations indirectly[15]. It is not commonly observed, although it becomes

[15] following Edward V Huntington, *Trans. Amer. Math. Soc.*, 5 (1904) 288–309.

evident when we consider it, that with a set of only two initials, a direct proof of their independence is always available, and I give such a proof in the text.

An independence proof may be properly considered as an incompleteness proof of the calculus with the missing initial.

Chapter 11

The question of whether or not functions of themselves are allowable has been discussed at wearisome length by many authorities [cf 8] since *Principia mathematica* was published. The Whitehead-Russell argument for disallowing them is well known. It is the subject of a number of comments by Wittgenstein [4, propositions 5.241 sq]. (I use the Pears-McGuinness translation for what follows.)

An operation, says Wittgenstein, is not the mark of a form, but of a relation between forms. Wittgenstein here sees what I call the mark of distinction between states, which he calls forms, and also sees its connexion with the idea of operation. He then remarks [5.251] that

> a function cannot be its own argument, whereas an operation can take one of its own results as its base.

This applies only, in the strict sense, to single-valued functions. If we allow inverse and implicit functions, then the assertion above is untrue. A function of a variable, in the wider meaning with which it is defined in this chapter, is the result of a possible set of operations on the variable. Thus if an operation can take its own result as a base, the function determined by this operation can be its own argument.

I shall proceed, in the light of this relaxation, to examine in some detail the analogy between Boolean equations and those of an ordinary numerical algebra.

Boole maintained[16] that the equation with which he defined what he called the law of duality, notably

$$x^2 = x,$$

[16] George Boole, *An investigation of the laws of thought*, Cambridge, 1854, p 50.

is of the second degree. So it is, as stated, but by it he determines that, in his notation, all equations of degree >1 shall be reduced to the first degree. In other words, it is an equation of the second degree only at the descriptive level, not in the algebra itself.

The spuriousness of its alleged degree, considered in the algebra itself, is revealed by Boole's assertion in a footnote [p 50] that an equation of the third degree has no interpretation in his algebra. It has, as we shall presently see, but Boole appears at this point to have been overcome by his notation, which uses numerical forms for an algebra which is essentially non-numerical.

Boole's equation

$$x^2 = x$$

is an analogue, in the primary algebra, of

$$aa = a.$$

This, as we see, is an equation of the first degree, being expressible without subversion. The real form of the analogy with a numerical algebra may be illustrated as follows.

Suppose

$$px^2 + qx + r = 0$$

where p, q, r may stand for rational numbers. We can re-express this equation in the form

F1 $$x^2 + ax + b = 0$$

by calling $q/p = a$ and $r/p = b$, and it may then be further transposed into

$$x = -a + \frac{-b}{x}$$

F2 $$= -a + \cfrac{-b}{-a + \cfrac{-b}{-a + \cfrac{-b}{\dots}}}\,.$$

In a Boolean algebra we are properly denied the mode of F1, but permitted the mode of F2, which is either continuous or, if we want to see it so, subversive. Thus an equation of any degree is both constructible and meaningful in a Boolean algebra, although not necessarily in the primary form of it. To reach a higher degree, all we need to do is to add a distinct subversion. The two modulator equations at the end of the chapter are both of degree >2. They were first developed in 1961, in collaboration with Mr D J Spencer-Brown, for special-purpose computer circuits. Such equations undertake an excursion to a higher order of infinity, and, although still expressible in subversive form, they cannot be represented in continuous form on a plane.

The circuits represented by these equations, the latter being presently in use by British Railways, comprise, as far as we know, a first application of each of two inventions, notably the first construction of a device which counts entirely by 'logic' (i.e. with switches only, and with no artificial time delays such as electrical condensers) and, in addition, the first use, in a switching circuit, of imaginary Boolean values in the course of the construction of a real answer. This latter might in fact be the first use of such imaginary values for any purpose, although it is my guess that Fermat (who was apparently too excellent a mathematician to make a false claim to a proof) used them in the proof of his great theorem, hence the 'truly remarkable' nature of his proof, as well as its length.

The fact that imaginary values *can* be used to reason towards a real and certain answer, coupled with the fact that they *are not* so used in mathematical reasoning today, and also coupled with the fact that certain equations plainly *cannot* be solved without the use of imaginary values, means that *there* **must** *be mathematical statements* (whose truth or untruth is in fact perfectly decidable) *which cannot be decided by the methods of reasoning to which we have hitherto restricted ourselves.*

Generally speaking, if we confine our reasoning to an interpretation of Boolean equations of the first degree only, we should *expect* to find theorems which will always defy decision, and the fact that we do seem to find such theorems in common arithmetic may serve, here, as a practical confirmation of this obvious prediction. To confirm it theoretically, we need only to prove (1) that such theorems *cannot* be decided by reasoning of the first degree, and (2) that they *can* be decided by reasoning of a higher degree. (2) would of course be proved by providing such a proof of one of these theorems.

I may say that I believe that at least one such theorem will shortly be decided by the methods outlined in the text. In other words, I believe that I have reduced their decision to a technical problem which is well within the capacity of an ordinary mathematician who is prepared, and who has the patronage or other means, to undertake the labour.

Any evenly subverted equation of the second degree might be called, alternatively, evenly informed. We can see it over a sub-version (turning under) of the surface upon which it is written, or alternatively, as an in-formation (formation within) of what it expresses.

Such an expression is thus informed in the sense of having its own form within it, and at the same time informed in the sense of remembering what has happened to it in the past.

We need not suppose that this is exactly how memory happens in an animal, but there are certainly memories, so-called, constructed this way in electronic computers, and engineers have constructed such in-formed memories with magnetic relays for the greater part of the present century.

We may perhaps look upon such memory, in this simplified in-formation, as a precursor of the more complicated and varied forms of memory and information in man and the higher animals. We can also regard other manifestions of the classical forms of physical or biological science in the same spirit.

Thus we do not imagine the wave train emitted by an excited finite echelon to be exactly like the wave train emitted from an excited physical particle. For one thing the wave form from an echelon is square, and for another it is emitted without energy. (We should need, I guess, to make at least one more departure from the form before arriving at a conception of energy on these lines.) What we see in the forms of expression at this stage, although recognizable, might be considered as simplified precursors of what we take, in physical science, to be the real thing. Even so, their accuracy and coverage is striking. For example, if, instead of considering the wave train emitted by the expression in Figure 4, we consider the expression itself, in its quiescent state, we see that it is composed of standing waves. If, therefore, we shoot such an expression through its own representative space, it will, upon passing a given point, be observable at that point as a simple oscillation with a frequency proportional to the velocity of its passage. We have thus already arrived, even at this stage, at a remarkable and striking precursor of the wave properties of material particles.

We may look upon such manifestations as the formal seeds, the existential forerunners, of what must, in a less central state, under less certain conditions, come about. There is a tendency, especially today, to regard existence as the source of reality, and thus as a central concept. But as soon as it is formally examined (cf Appendix 2), existence* is seen to be highly peripheral and, as such, especially corrupt (in the formal sense) and vulnerable. The concept of truth is more central, although still recognizably peripheral. If the weakness of present-day science is that it centres round existence, the weakness of present-day logic is that it centres round truth.

Throughout the essay, we find no need of the concept of truth, apart from two avoidable appearances (true = open to proof) in the descriptive context. At no point, to say the least, is it a necessary inhabitant of the calculating forms. These forms are thus not only precursors of existence, they are also precursors of truth.

It is, I am afraid, the intellectual block which most of us come up against at the points where, to experience the world clearly, we must abandon existence to truth, truth to indication, indication to form, and form to void, that has so held up the development of logic and its mathematics.

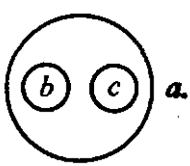
What status, then, does logic bear in relation with mathematics? We may anticipate, for a moment, Appendix 2, from

* *ex* = out, *stare* = stand. Thus to exist may be considered as to stand outside, to be exiled.

which we see that the arguments we used to justify the calculating forms (e.g. in the proofs of theorems) *can themselves be justified by putting them in the form of the calculus.* The process of justification can be thus seen to feed upon itself, and this may comprise the strongest reason against believing that the codification of a proof procedure lends evidential support to the proofs in it. All it does is provide them with coherence. A theorem is no more proved by logic and computation than a sonnet is written by grammar and rhetoric, or than a sonata is composed by harmony and counterpoint, or a picture painted by balance and perspective. Logic and computation, grammar and rhetoric, harmony and counterpoint, balance and perspective, can be seen in the work *after* it is created, but these forms are, in the final analysis, parasitic on, they have no existence apart from, the creativity of the work itself. Thus the relation of logic to mathematics is seen to be that of an applied science to its pure ground, and all applied science is seen as drawing sustenance from a process of creation with which it can combine to give structure, but which it cannot appropriate.

Chapter 12

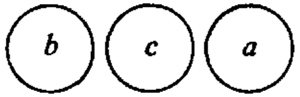
Let us imagine that, instead of writing on a plane surface, we are writing on the surface of the Earth. Ignoring rabbit holes, etc, we may take it to be a surface of genus 0. Suppose we write



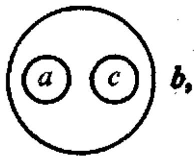
To make it readable from another planet, we write it large. Suppose we draw the outer bracket round the Equator, and make the brackets containing *b* and *c* follow the coastlines of Australia and the South Island of New Zealand respectively.

Above is how the expression will appear from somewhere in the Northern Hemisphere, say London. But let us travel.

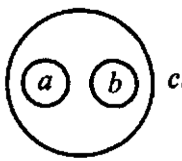
Arriving at Cape Town we see



Sailing on to Melbourne, we see



and proceeding from there to Christchurch, we see



These four expressions are distinct and not equivalent. Thus it is evidently not enough merely to write down an expression, even on a surface of genus 0, and expect it to be understood. We must also indicate where the observer is supposed to be standing in relation to the expression. Writing on a plane, the ambiguity is not apparent because we tend to see the expression from outside of the outermost bracket. When it is written on the surface of a sphere, there may be no means of telling which of the brackets is supposed to be outermost. In such a case, to make an expression meaningful, we must add to it an indicator to present a place from which the observer is invited to regard it.

We observe in the third experiment an alternative way (although here less powerful) of using the principle of relevance. By the normal use of the principle we could obliterate the additional markings (since every state is identically marked) and arrive at the single circle in one step, whereas in the experiment we take the weaker course of obliterating the line of distinction between the markings, and then need one more step to reach the single circle.

Note that both of these ways of simplification are *different* from the methods of cancellation and condensation adopted for the calculus, although arising from, and thus not inconsistent with, them. From the experiment we begin to see in fact how all the constellar principles by which we navigate our journeys out from and in to the form spring from the ultimate reducibility of numbers and voidability of relations. It is only by arresting or fixing the use of these principles at some stage that we manage to maintain a universe in any form at all, and our understanding of such a universe comes not from discovering its present appearance, but in remembering what we originally did to bring it about.

In this way the calculus itself can be realized as a direct recollection. As we left the central state of the form, proceeding outwards and imagewise towards the peripheral condition of existence, we saw how the laws of calling and crossing, which set the stage of our journey through representative space, became fixed stars in the familiar play of time. Our projected hopes and fears of their ultimate atonement, which we called theorems, became their supporting cast. In the end, as we re-enter the form, they are all justified and expended. They were needed only as long as they were doubted. When they cannot be doubted, they can be discarded.

Returning, briefly, to the idea of existential precursors, we see that if we accept their form as endogenous to the less primitive structure identified, in present-day science, with reality, we cannot escape the inference that what is commonly now regarded as real consists, in its very presence, merely of tokens or expressions. And since tokens or expressions are considered to be *of* some (other) substratum, so the universe itself, as we know it, may be considered to be an expression of a reality other than itself.

Let us then consider, for a moment, the world as described by the physicist. It consists of a number of fundamental particles which, if shot through their own space, appear as waves, and are thus (as in Chapter 11), of the same laminated structure as pearls or onions, and other wave forms called electromagnetic which it is convenient, by Occam's razor, to consider as travelling through space with a standard velocity. All these appear bound by certain natural laws which indicate the form of their relationship.

Now the physicist himself, who describes all this, is, in his own account, himself constructed of it. He is, in short, made of a conglomeration of the very particulars he describes, no more, no less, bound together by and obeying such general laws as he himself has managed to find and to record.

Thus we cannot escape the fact that the world we know is constructed in order (and thus in such a way as to be able) to see itself.

This is indeed amazing.

Not so much in view of what it sees, although this may appear fantastic enough, but in respect of the fact that it *can* see *at all*.

But *in order* to do so, evidently it must first cut itself up into at least one state which sees, and at least one other state which is seen. In this severed and mutilated condition, whatever it sees is *only partially* itself. We may take it that the world undoubtedly is itself (i.e. is indistinct from itself), but, in any attempt to see itself as an object, it must, equally undoubtedly, act* so as to make itself distinct from, and therefore false to, itself. In this condition it will always partially elude itself.

It seems hard to find an acceptable answer to the question of how or why the world conceives a desire, and discovers an ability, to see itself, and appears to suffer the process. That it does so is sometimes called the original mystery. Perhaps, in view of *the form* in which *we* presently *take* ourselves *to exist*, the mystery *arises from* our insistence on *framing* a question where there is, in reality, *nothing* to question. However it may appear, if such desire, ability, and sufferance be granted, the state or condition that arises as an outcome is, according

* Cf ἀγωνιστής = actor, antagonist. We may note the identity of action with agony.

to the laws here formulated, absolutely unavoidable. In this respect, at least, there is no mystery. We, as universal representatives, *can* record universal law far enough to say

> and so on, and so on you will eventually construct the universe, in every detail and potentiality, as you know it now; but then, again, what you will construct will not be all, for by the time you will have reached what now is, the universe will have expanded into a new order to contain what will then be.

In this sense, in respect of its own information, the universe *must* expand to escape the telescopes through which we, who are it, are trying to capture it, which is us. The snake eats itself, the dog chases its tail.

Thus the world, when ever it appears as a physical universe*, must always seem to us, its representatives, to be playing a kind of hide-and-seek with itself. What is revealed will be concealed, but what is concealed will again be revealed. And since we ourselves represent it, this occultation will be apparent in our life in general, and in our mathematics in particular. What I try to show, in the final chapter, is the fact that we really knew all along that the two axioms by which we set our course were mutually permissive and agreeable. At a certain stage in the argument, we somehow cleverly obscured this knowledge from ourselves, in order that we might then navigate ourselves through a journey of rediscovery, consisting in a series of justifications and proofs with the purpose of again rendering, to ourselves, irrefutable evidence of what we already knew.

Coming across it thus again, in the light of what we had to do to render it acceptable, we see that our journey was, in its preconception, unnecessary, although its formal course, once we had set out upon it, was inevitable.

* *unus* = one, *vertere* = turn. Any given (or captivated) universe is what is seen as the result of a making of one turn, and thus *is the appearance* of any first distinction, and only a minor aspect of all being, apparent and non-apparent. Its particularity is the price we pay for its visibility.

INDEX OF REFERENCES

Note. In context, a page reference is confined to what is of particular interest to the discussion. In this index it is expanded to include the whole work.

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About the Author

Born in Lincolnshire, England, G Spencer-Brown was educated at Trinity College, Cambridge, where he became personally acquainted with Russell and Wittgenstein. After graduating, he was elected a research lecturer of Christ Church, Oxford, and here made his first contributions to the mathematical theory of probability.

In 1964, on Russell's recommendation, he became lecturer in formal mathematics at the University of London, and joined the Cambridge University department of pure mathematics in 1969. He was visiting professor at the University of Western Australia in 1976, and at Stanford University in 1977.

Apart from the non-numerical calculi he himself devised, his chief mathematical interest has been the theory of numbers, and in particular the determination of primality.

He is also a chess half-blue, holds two world records as a glider pilot, and was a sports correspondent to the *Daily Express*.

In 1847 George Boole devised an algebra for solving problems in logic, but it was over a century later that Professor G. Spencer-Brown finally succeeded in uncovering the elegant and fascinating arithmetic that forms the basis of Boolean algebra (which has proved to have applications beyond the field of logic--for example, in the design of computer circuits). This new calculus is presented in the now-famous *Laws of Form*. It would be hard to exaggerate the significance of this work, or the facilities it offers to readers of all kinds, whether or not they are acquainted with present techniques of mathematical inquiry. This revised edition of *Laws of Form* includes a new Preface by the author.

"Bertrand Russell said that 'in this book Mr. Spencer-Brown has succeeded in doing what is very rare indeed. He has revealed a new calculus of great power and simplicity. I congratulate him.' This is generous, but I believe it will eventually be recognized that Mr. Spencer-Brown has done more than that. He has not invented an arbitrary new calculus, but that particular calculus which can let us see deeper into the nature of mathematics. Indeed I still consider, on re-examining this book after a two-year interval, that it is a work of genius."

—Lancelot Law Whyte, *British Journal of the Philosophy of Science*

"I suspect I am reviewing a work of genius. We are introduced to an algebra of the utmost simplicity, which a child (if he were thus sophisticated) could understand, and to a notation of great beauty, because it conveys what it says."

—Stafford Beer, *Nature*

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E.P. Dutton
2 Park Avenue